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LETTER TO THE EDITOR

h analogue of Newton's binomial formula

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Abstract. In this letter, the h analogue of Newton's binomial formula is obtained in the h deformed quantum plane which does not have any q analogue. For $h = 0$, this is just the usual one, as it should be. Furthermore, the binomial coefficients reduce to $\frac{n!}{(n-k)!}$ for $h = 1$. Some properties of the h binomial coefficients are also given. Finally, it is hoped that such results will contribute to an introduction of the h analogue of the well known functions, h special functions and h deformed analysis.

The study of q analysis appeared in the literature a long time ago [1]. In particular, a q analogue of Newton's formula, well known functions like the q exponential, q logarithm, etc, and the special function arena's [1, 5, 6] have been introduced and studied intensively.

A q analogue of these was obtained by taking q commuting variables, x, y, satisfying the relation $xy = qyx$, i.e. (x, y) belong to the Manin plane.

In this letter, I will take another direction by introducing the analogue of Newton's formula in the h deformed quantum plane [8, 7] (i.e. h Newton binomial formula). As far as I know, such an h analogue did not exist in the literature until now and the result will, in the future, permit the introduction of the h analogue of well known functions, h special functions and h deformed analysis.

Newton's binomial formula is defined as follows:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} \tag{1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and it is understood here that the coordinate variables x and y commute, i.e. $xy = yx$.

A q analogue of (1) for the q commuting coordinates x and y satisfying $xy = qyx$ was first stated by Rothe, although its special cases were known to Euler, see [3], found again by Schützenberger [2], and has subsequently been rediscovered many times [4].

A q analogue of (1) becomes

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} \tag{2}$$

where the q binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

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with

$$(a;q)_k = (1-a)(1-qa)\ldots(1-q^{k-1}a) \qquad a \in \mathbb{C},\ k \in \mathbb{N}.$$

Now consider Manin's q plane $x'y' = qy'x'$. By the following linear transformation (see [8] and references therein)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Manin's q plane changes to $xy - qyx = hy^2$ which for $q = 1$ gives the h deformed plane

$$xy = yx + hy^2. \tag{3}$$

Even though the linear transformation is singular for $q = 1$, the resulting quantum plane is well defined.

Proposition 1. Let x and y be coordinate variables satisfying (3), then the following identities are true:

$$\begin{aligned} x^k y &= \sum_{r=0}^{k} \frac{k!}{(k-r)!} h^r y^{r+1} x^{k-r} \\ xy^k &= y^k x + khy^{k+1}. \end{aligned} \tag{4}$$

These identities are easily proved by successive use of (3).

Proposition 2 (h binomial formula). Let x and y be coordinate variables satisfying (3), then we have

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_h y^k x^{n-k} \tag{5}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_h$ are the h binomial coefficients given as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_h = \begin{pmatrix} n \\ k \end{pmatrix} h^k (h^{-1})_k \tag{6}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_h = 1$ and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the shifted factorial.

Proof. We will prove this proposition by recurrence. Indeed for $n = 1, 2$, it is verified.

Suppose now that the formula is true for $n-1$, which means

$$(x+y)^{n-1} = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h y^k x^{n-1-k}$$

with $\begin{bmatrix} n-1 \\ 0 \end{bmatrix}_h = 1$.

To show this for n, let us first consider the following expansion:

$$(x+y)^n = \sum_{k=0}^{n} C_{n,k} y^k x^{n-k}$$

where $C_{n,k}$ are coefficients depending on h.

Then, we have

$$
\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h y^k x^{n-1-k} \\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h x y^k x^{n-1-k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h y^{k+1} x^{n-1-k}.
\end{aligned}
$$

Using the result of the first proposition, we obtain

$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h y^k x^{n-k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h (1+kh) y^{k+1} x^{n-1-k} \\
&= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_h y^k x^{n-k} + \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_h (1+(k-1)h) y^k x^{n-k}
\end{aligned}
$$

which yields respectively

$$
\begin{aligned}
C_{n,0} &= \begin{bmatrix} n-1 \\ 0 \end{bmatrix}_h = 1 \\
C_{n,k} &= \begin{bmatrix} n-1 \\ k \end{bmatrix}_h + (1+(k-1)h) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_h = \begin{bmatrix} n \\ k \end{bmatrix}_h \\
C_{n,n} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_h (1+(n-1)h) = \begin{bmatrix} n \\ n \end{bmatrix}_h .
\end{aligned}
$$

□

Moreover, the h binomial coefficients obey the following properties:

$$
\begin{bmatrix} n \\ k \end{bmatrix}_h + (1+(k-1)h) \begin{bmatrix} n \\ k-1 \end{bmatrix}_h = \begin{bmatrix} n+1 \\ k \end{bmatrix}_h \tag{7}
$$

and

$$
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_h = \frac{n+1}{k+1}(1+kh) \begin{bmatrix} n \\ k \end{bmatrix}_h . \tag{8}
$$

In fact, these properties follow from the well known relations of the classical binomial coefficients:

$$
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
$$

and

$$
\binom{n+1}{k} = \frac{n+1}{k} \binom{n}{k-1}
$$

upon using $(a)_k = (a+k-1)(a)_{k-1}$, which means that (7) and (8) are just a consequence of the known properties of the classical coefficients and the shifted factorial.

Now, we make the following remarks. First, for $h = 0$ the Newton's binomial formula is just the usual one for commuting variables $xy = yx$, as it should be.

Second, for $h = 1$ the $h = 1$-binomial coefficients are

$$
\begin{bmatrix} n \\ k \end{bmatrix}_{h=1} = \frac{n!}{(n-k)!} \tag{9}
$$

and therefore the $h = 1$-analogue of Newton's binomial formula becomes

$$(x + y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} y^k x^{n-k} \tag{10}$$

provided that $xy = yx + y^2$.

To conclude, we see that the h analogue of Newton's formula in the h deformed plane has no q analogue. It seems from the structures of the h binomial coefficients that the h deformed plane is somewhat 'more classical' than the q deformed plane.

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